

# ALMOST NONNEGATIVE CURVATURE ON SOME FAKE $\mathbb{R}P^6$ S AND $\mathbb{R}P^{14}$ S

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**ABSTRACT.** We apply the lifting theorem of Searle and the second author to put metrics of almost nonnegative curvature on the fake  $\mathbb{R}P^6$ 's of Hirsch and Milnor and on the analogous fake  $\mathbb{R}P^{14}$ 's.

One of the great unsolved problems of Riemannian geometry is to determine the structure of collapse with a lower curvature bound. An apparently simpler, but still intractable problem, is to determine which closed manifolds collapse to a point with a lower curvature bound. Such manifolds are called almost nonnegatively curved. Here we construct almost nonnegative curvature on some fake  $\mathbb{R}P^6$ s and  $\mathbb{R}P^{14}$ s.

**Theorem A.** *The Hirsch-Milnor fake  $\mathbb{R}P^6$ s and the analogous fake  $\mathbb{R}P^{14}$ s admit Riemannian metrics that simultaneously have almost nonnegative sectional curvature and positive Ricci curvature.*

**Remark.** *By considering cohomogeneity one actions on Brieskorn varieties, Schwachhöfer and Tuschmann observed in [15] that in each odd dimension of the form,  $4k + 1$ , there are at least  $4^k$  oriented diffeomorphism types of homotopy  $\mathbb{R}P^{4k+1}$ s that admit metrics that simultaneously have positive Ricci curvature and almost nonnegative sectional curvature.*

The Hirsch-Milnor fake  $\mathbb{R}P^6$ s are quotients of free involutions on the images of embeddings  $\iota$  of the standard 6-sphere,  $\mathbb{S}^6$ , into some of the Milnor exotic 7-spheres,  $\Sigma_k^7$  ([12], [14]). Our proof begins with the observation that the  $SO(3)$ -actions that Davis constructed on the  $\Sigma_k^7$ s in [5] leave these Hirsch-Milnor  $S^6$ s invariant and commute with the Hirsch-Milnor free involution. Next we compare the Hirsch-Milnor/Davis  $(SO(3) \times \mathbb{Z}_2)$ -action on  $\iota(S^6) \subset \Sigma_k^7$  with a very similar linear action of  $(SO(3) \times \mathbb{Z}_2)$  on  $\mathbb{S}^6 \subset \mathbb{R}^7$  and apply the following lifting result of Searle and the second author.

**Theorem B.** (See Proposition 8.1 and Theorems B and C in [17]) *Let  $(M_e, G)$  and  $(M_s, G)$  be smooth, compact,  $n$ -dimensional  $G$ -manifolds with  $G$  a compact Lie group. Suppose that the orbit spaces  $M_e/G$  and  $M_s/G$  are equivalent, and  $M_s/G$  has almost nonnegative curvature. Then  $M_e$  admits a  $G$ -invariant family of metrics that has almost nonnegative sectional curvature. Moreover, if the principal orbits of  $(M_e, G)$  have finite fundamental group and the quotient of the principal orbits of  $M_s$  has Ricci curvature  $\geq 1$ , then every metric in the almost nonnegatively curved family on  $M_e$  can be chosen to also have positive Ricci curvature.*

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We emphasize that to apply Theorem B,  $M_s/G$  need not be a Riemannian manifold, but since  $M_s$  is compact,  $M_s/G$  is an Alexandrov space with curvature bounded from below. The meaning of almost nonnegative curvature for Alexandrov spaces is as follows.

**Definition.** We say that a sequence of Alexandrov spaces  $\{(X, \text{dist}_\alpha)\}_\alpha$  is almost nonnegatively curved if and only if there is a  $D > 0$  so that

$$\sec(X, g_\alpha) \geq -\frac{1}{\alpha} \text{ and } \text{Diam}(X, g_\alpha) \leq D,$$

or equivalently, after a rescaling,  $X$  collapses to a point with a uniform lower curvature bound.

The following is the precise notion of equivalence of orbit spaces required by the hypotheses of Theorem B.

**Definition.** Suppose  $G$  acts on  $M_e$  and on  $M_s$ . We say that the orbit spaces  $M_e/G$  and  $M_s/G$  are equivalent if and only if there is a strata-preserving homeomorphism  $\Phi : M_e/G \rightarrow M_s/G$  whose restriction to each stratum is a diffeomorphism with the following property:

Let  $\pi_s : M_s \rightarrow M_s/G$  and  $\pi_e : M_e \rightarrow M_e/G$  be the quotient maps. If  $\mathcal{S} \subset M_e$  is a stratum, then for any  $x_e \in \mathcal{S}$  and any  $x_s \in \pi_s^{-1}(\Phi(\pi_e(x_e)))$ , the action of  $G_{x_e}$  on  $\nu(\mathcal{S})_{x_e}$  is linearly equivalent to the action of  $G_{x_s}$  on  $\nu(\mathcal{S})_{x_s}$ . Here  $G_x$  is the isotropy subgroup at  $x$  and  $\nu(\mathcal{S})_x$  is the normal space to  $\mathcal{S}$  at  $x$ .

To construct the metrics on the fake  $\mathbb{R}P^6$ s of Theorem A, we apply Theorem B with  $G = (SO(3) \times \mathbb{Z}_2)$ .  $M_e$  will be the Hirsch-Milnor embedded image of  $\mathbb{S}^6$  in  $\Sigma_k^7$ , and  $M_s$  will be  $\mathbb{S}^6$  with the following  $(SO(3) \times \mathbb{Z}_2)$ -action: View  $\mathbb{S}^6$  as the unit sphere in  $\mathbb{H} \oplus \text{Im } \mathbb{H}$ , where  $\mathbb{H}$  stands for the quaternions, and let  $SO(3) \times \mathbb{Z}_2$  act on  $\mathbb{S}^6 \subset \mathbb{H} \oplus \text{Im } \mathbb{H}$  via

$$\begin{aligned} SO(3) \times \mathbb{Z}_2 \times \mathbb{S}^6 &\rightarrow \mathbb{S}^6 \\ (g, \pm, (a, c)) &\mapsto \pm(g(a), g(c)). \end{aligned} \tag{0.0.1}$$

Here the  $SO(3)$ -action on the  $\mathbb{H}$ -factor is the direct sum of the standard action of  $SO(3)$  on  $\text{Im } \mathbb{H}$  with the trivial action on  $\text{Re } (\mathbb{H})$ .

Since quotient maps of isometric group actions preserve lower curvature bounds,  $\mathbb{S}^6/(SO(3) \times \mathbb{Z}_2)$  has curvature  $\geq 1$  ([4]). Thus to construct the metrics on the fake  $\mathbb{R}P^6$ s of Theorem A, it suffices to combine Theorem B with the following result.

**Lemma C.** The orbit space of the Hirsch-Milnor/Davis action of  $SO(3) \times \mathbb{Z}_2$  on  $\iota(\mathbb{S}^6) \subset \Sigma_k^7$  is equivalent to the orbit space of the linear action (0.0.1) on  $\mathbb{S}^6$ .

Our metrics on fake  $\mathbb{R}P^{14}$ s are octonionic analogs of our metrics on fake  $\mathbb{R}P^6$ s. The analogy begins with Shimada's observation that Milnor's proof of the total spaces of certain  $\mathbb{S}^3$ -bundles over  $\mathbb{S}^4$  being exotic spheres also applies to certain  $\mathbb{S}^7$ -bundles over  $\mathbb{S}^8$  ([18]). Davis's construction of the  $SO(3)$ -actions on  $\Sigma_k^7$ s is based on the fact that  $SO(3)$  is the group of automorphisms of  $\mathbb{H}$ . Exploiting the fact that  $G_2$  is the group automorphisms of the octonions,  $\mathbb{O}$ , Davis constructs analogous  $G_2$  actions on Shimada's exotic  $\Sigma_k^{15}$ s. By applying a result of Brumfiel ([3]), we will see that the Hirsch and Milnor construction of fake  $\mathbb{R}P^6$ s as quotients of  $\iota(\mathbb{S}^6) \subset \Sigma_k^7$  also works to construct fake  $\mathbb{R}P^{14}$ s as quotients of  $\iota(\mathbb{S}^{14}) \subset \Sigma_k^{15}$ . Thus to construct the fake  $\mathbb{R}P^{14}$ s of Theorem A, it suffices to show the following.

**Lemma D.** *The orbit space of the Hirsch-Milnor/Davis action of  $G_2 \times \mathbb{Z}_2$  on  $\iota(\mathbb{S}^{14}) \subset \Sigma_k^{15}$  is equivalent to the orbit space of the following linear action of  $G_2 \times \mathbb{Z}_2$  on  $\mathbb{S}^{14} \subset \mathbb{O} \oplus \text{Im } \mathbb{O}$ ,*

$$\begin{aligned} G_2 \times \mathbb{Z}_2 \times \mathbb{S}^{14} &\longrightarrow \mathbb{S}^{14} \\ (g, \pm, (a, c)) &\mapsto \pm(g(a), g(c)). \end{aligned} \quad (0.0.2)$$

In Section 1, we review the construction of the Hirsch-Milnor and Davis actions and explain why the Hirsch-Milnor construction works in the Octonionic case. In Section 2, we prove Lemmas C and D and hence Theorem A, and in Section 3, we make some concluding remarks. We refer the reader to page 185 of [2] for a description of how  $G_2$  acts as automorphisms of the Octonions.

**Remark.** *Explicit formulas for exotic involutions on  $\mathbb{S}^6$  and  $\mathbb{S}^{14}$  are given in ([1]), where it is shown, on pages 13–17, that the corresponding fake  $\mathbb{R}P^6$  is diffeomorphic to the Hirsch-Milnor  $\mathbb{R}P^6$  that corresponds to  $\Sigma_3^7$ .*

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## 1. HOW TO CONSTRUCT EXOTIC REAL PROJECTIVE SPACES

In this section, we review Milnor spheres, the Hirsch-Milnor construction, and the Davis actions. We then explain how the Hirsch-Milnor argument gives fake  $\mathbb{R}P^{14}$ s.

To construct the Milnor spheres, we write  $\Lambda$  for  $\mathbb{H}$  or  $\mathbb{O}$  and  $b$  for the real dimension of  $\Lambda$ . To get a  $\mathbb{S}^{b-1}$ -bundle over  $\mathbb{S}^b$  with structure group  $SO(b)$ ,  $(E_{h,j}, p_{h,j})$ , we glue two copies of  $\Lambda \times \mathbb{S}^{b-1}$  together via

$$\begin{aligned} \Phi_{h,j} &: \Lambda \setminus \{0\} \times \mathbb{S}^{b-1} \longrightarrow \Lambda \setminus \{0\} \times \mathbb{S}^{b-1} \\ \Phi_{h,j} &: (u, q) \mapsto \left( \frac{u}{|u|^2}, \left( \frac{u}{|u|} \right)^h q \left( \frac{u}{|u|} \right)^j \right). \end{aligned} \quad (1.0.3)$$

To define the projection  $p_{h,j} : E_{h,j} \longrightarrow \mathbb{S}^b$ , we think of  $\mathbb{S}^b$  as obtained by gluing together two copies of  $\Lambda$  along  $\Lambda \setminus \{0\}$  via  $u \mapsto \frac{u}{|u|^2}$ .  $p_{h,j}$  is then defined to be the projection to either copy of  $\Lambda$ .

When  $h + j = \pm 1$ , the smooth function

$$f : (u, q) \mapsto \frac{\text{Re}(q)}{\sqrt{1 + |u|^2}} = \frac{\text{Re}(vr^{-1})}{\sqrt{1 + |v|^2}}$$

is regular except at  $(u, q) = (0, \pm 1)$ . Hence,  $E_{h,j}$  is homeomorphic to  $\mathbb{S}^{2b-1}$  if  $h + j = \pm 1$ , and a Mayer-Vietoris argument shows that  $E_{h,j}$  is not homeomorphic to  $\mathbb{S}^{2b-1}$  if  $h + j \neq \pm 1$ . Since  $f(0, \pm 1) = \pm 1$ , it also follows that  $f^{-1}(0)$  is diffeomorphic to  $\mathbb{S}^{2b-2}$ .

From now on we assume that

$$h + j = 1, \quad (1.0.4)$$

and we set

$$k = h - j. \quad (1.0.5)$$

So

$$k = 2h - 1.$$

For simplicity, we will write  $\Sigma_k^{2b-1}$  for  $E_{h,j}$  and  $\Phi_k$  for  $\Phi_{h,j}$ , and set

$$S_k^{2b-2} \equiv f^{-1}(0).$$

The Hirsch-Milnor construction ([12]) begins with the observation that the involution

$$\begin{aligned} T &: \Lambda \times \mathbb{S}^{b-1} \longrightarrow \Lambda \times \mathbb{S}^{b-1} \\ T &: (u, q) \mapsto (u, -q) \end{aligned}$$

induces a well-defined free involution of  $\Sigma_k^{2b-1}$ . Moreover,  $T$  leaves  $S_k^{2b-2}$  invariant. Lemma 3 of [12] says that the quotient of any fixed point free involution on  $\mathbb{S}^n$  is homotopy equivalent to  $\mathbb{R}P^n$ . In particular, all of our spaces

$$P_k^{2b-2} \equiv S_k^{2b-2}/T$$

are homotopy equivalent to  $\mathbb{R}P^{2b-2}$ . Hirsch and Milnor then show that when  $b = 4$ ,  $P_k^6$  is not diffeomorphic to  $\mathbb{R}P^6$ , provided  $\Sigma_k^7$  is an odd element of  $\Theta_7$ , the group of oriented diffeomorphism classes of differential structures on  $\mathbb{S}^7$ . According to pages 102 and 103 of [6], there are 16 oriented diffeomorphism classes among the  $\Sigma_k^7$ s, and among these, 8 are odd elements of  $\Theta_7$ .

To understand how this works octonionically, we let  $\Theta_{15}$  be the group of oriented diffeomorphism classes of differential structures on  $\mathbb{S}^{15}$ , and we let  $bP_{16}$  be the set of the elements of  $\Theta_{15}$  that bound parallelizable manifolds. According to [13],  $bP_{16}$  is a cyclic subgroup of  $\Theta_{15}$  of order 8, 128 and index 2, and according to Theorem 1.3 in [3],  $\Theta_{15}$  is not cyclic. Thus

$$\begin{aligned} \Theta_{15} &\cong bP_{16} \oplus \mathbb{Z}_2 \\ &\cong \mathbb{Z}_{8,128} \oplus \mathbb{Z}_2. \end{aligned}$$

According to Wall ([19]), a homotopy sphere bounds a parallelizable manifold if and only if it bounds a 7-connected manifold. In particular, each of the  $\Sigma_k^{15}$ s is in  $bP_{16}$ .

According to pages 101–107 of [6],  $\Sigma_k^{15}$  represents an odd element of  $bP_{16}$  if and only if  $\frac{h(h-1)}{2}$  is odd, that is,  $h$  is congruent to 2 or 3 mod 4.

The Hirsch-Milnor argument, combined with the fact that  $\Theta_{15} \cong bP_{16} \oplus \mathbb{Z}_2$ , implies  $P_k^{14}$  is not diffeomorphic to  $\mathbb{R}P^{14}$ , if  $\Sigma_k^{15}$  is an odd element of  $bP_{16}$ .

We let

$$G^\Lambda \equiv \begin{cases} SO(3) & \text{when } \Lambda = \mathbb{H} \\ G_2 & \text{when } \Lambda = \mathbb{O}. \end{cases}$$

Davis observed that since  $G^\Lambda$  is the automorphism group of  $\Lambda$ , the diagonal action

$$\begin{aligned} G^\Lambda \times \Lambda \times \mathbb{S}^{b-1} &\longrightarrow \Lambda \times \mathbb{S}^{b-1} \\ g(u, v) &= (g(u), g(v)) \end{aligned} \quad (1.0.6)$$

induces a well-defined  $G^\Lambda$ -action on  $\Sigma_k^{2b-1}$  ([5]).

Next we observe that the Davis action leaves  $S_k^{2b-2} = f^{-1}(0)$  invariant and commutes with  $T$ , giving us the  $SO(3) \times \mathbb{Z}_2$  actions of Lemma C and the  $G_2 \times \mathbb{Z}_2$  actions of Lemma D.

## 2. IDENTIFYING THE ORBIT SPACES

In this section, we prove Lemmas C and D simultaneously and hence Theorem A. In Lemma 2.1 (below), we identify the quotient map for the standard  $G^\Lambda$ -action of  $\mathbb{S}^{2b-2}$ . In Lemma 2.2 (below), we identify the quotient map for the Davis action on  $S_k^{2b-2}$ . Then in Key Lemma 2.3, we show that the two  $G^\Lambda$  quotients are the same. It is then a simple matter to identify the two  $G^\Lambda \times \mathbb{Z}_2$  quotient spaces with each other.

**Lemma 2.1.** *Let  $\mathbb{S}^{2b-2}$  be the unit sphere in  $\Lambda \oplus \text{Im}(\Lambda)$ , and let  $\langle, \rangle$  be the real dot product. The map*

$$\begin{aligned} Q_s &: \mathbb{S}^{2b-2} \longrightarrow Q_s(\mathbb{S}^{2b-2}) \subsetneq \mathbb{R}^3 \\ \begin{pmatrix} a \\ c \end{pmatrix} &\longmapsto (|a|, \text{Re } a, \langle \text{Im } a, \text{Im } c \rangle) \end{aligned}$$

has the following properties.

1. The fibers of  $Q_s$  coincide with the orbits of the  $G^\Lambda$  action

$$\begin{aligned} G^\Lambda \times \mathbb{S}^{2b-2} &\longrightarrow \mathbb{S}^{2b-2} \\ (g, (a, c)) &\mapsto (g(a), g(c)). \end{aligned}$$

2. The image of  $Q_s$  is  $Q_s(\mathbb{S}^{2b-2}) =$

$$\left\{ (x, y, z) \mid x \in [0, 1], y \in [-x, x], z \in \left[ -\sqrt{(x^2 - y^2)(1 - x^2)}, \sqrt{(x^2 - y^2)(1 - x^2)} \right] \right\}.$$

3. The principal orbits are mapped to the interior of  $Q_s(\mathbb{S}^{2b-2})$ . The fixed points are mapped to  $(1, 1, 0)$  and  $(1, -1, 0)$ , and the other orbits are mapped to  $\partial Q_s(\mathbb{S}^{2b-2}) \setminus \{(1, 1, 0), (1, -1, 0)\}$ .

*Proof.* Part 2 follows from the observations that

$$\begin{aligned} |a| &\in [0, 1], \\ \text{Re } a &\in [-|a|, |a|], \\ \langle \text{Im } a, \text{Im } c \rangle &\in [-|\text{Im}(a)| |\text{Im}(c)|, |\text{Im}(a)| |\text{Im}(c)|], \end{aligned}$$

and

$$|\text{Im}(a)| |\text{Im}(c)| \in \left[ 0, \sqrt{(|a|^2 - \text{Re}(a)^2)(1 - |a|^2)} \right].$$

Since the three quantities  $|a|$ ,  $\text{Re } a$ ,  $\langle \text{Im } a, \text{Im } c \rangle$  are invariant under  $G^\Lambda$ , each orbit of  $G^\Lambda$  is contained in a fiber of  $Q_s$ .

Conversely, if  $(a_1, c_1)$  and  $(a_2, c_2)$  satisfy  $Q_s(a_1, c_1) = Q_s(a_2, c_2)$ , then

$$\begin{aligned} |a_1| &= |a_2| \\ \text{Re}(a_1) &= \text{Re}(a_2), \text{ and} \\ \langle \text{Im } a_1, \text{Im } c_1 \rangle &= \langle \text{Im } a_2, \text{Im } c_2 \rangle. \end{aligned}$$

Together with  $\text{Re}(c_i) = 0$  and  $|a_i|^2 + |c_i|^2 = 1$ , this gives

$$\begin{aligned} |\text{Im}(a_1)| &= |\text{Im}(a_2)| \\ |\text{Im}(c_1)| &= |\text{Im}(c_2)|. \end{aligned}$$

Since we also have  $\langle \text{Im } a_1, \text{Im } c_1 \rangle = \langle \text{Im } a_2, \text{Im } c_2 \rangle$ , it follows that an element of  $G^\Lambda$  carries  $(a_1, c_1)$  to  $(a_2, c_2)$ . This completes the proof of Part 1.

To prove Part 3, we first note that the orbit of  $(a, c)$  is not principal if and only if

$$|\langle \text{Im } a, \text{Im } c \rangle| = |\text{Im } (a)| |\text{Im } (c)|,$$

and this is equivalent to  $Q_s(a, c) \in \partial Q_s(a, c)$ . So the principal orbits are mapped onto the interior of  $Q_s(\mathbb{S}^{2b-2})$ .

On the other hand, the fixed points are  $(\pm 1, 0)$  and  $Q_s(\pm 1, 0) = (1, \pm 1, 0)$  as claimed.  $\square$

Before proceeding, recall that we view

$$\Sigma_k^{2b-1} = (\Lambda \times \mathbb{S}^{b-1}) \cup_{\Phi_k} (\Lambda \times \mathbb{S}^{b-1}),$$

where  $\Phi_k$  is determined by Equations (1.0.3), (1.0.4), and (1.0.5). Combining this with the definition of  $S_k^{2b-2}$ , we have that

$$S_k^{2b-2} = U_1 \cup_{\Phi_k} U_2,$$

where

$$\begin{aligned} U_1 &\equiv \{ (u, q) \in \Lambda \times \mathbb{S}^{b-1} \mid \text{Re } (q) = 0 \} \text{ and} \\ U_2 &\equiv \{ (v, r) \in \Lambda \times \mathbb{S}^{b-1} \mid \text{Re } (vr^{-1}) = \text{Re } \bar{v}r = 0 \}. \end{aligned}$$

The quotient map of the  $G^\Lambda$ -action on  $S_k^{2b-2}$  has the following description.

**Lemma 2.2.** *Let  $\phi : R^n \rightarrow R$  be given by,  $\phi(v) = \frac{1}{\sqrt{1+|v|^2}}$ .*

*The map*

$$\begin{aligned} Q_k &: S_k^{2b-2} \rightarrow Q_k(S_k^{2b-2}) \subsetneq \mathbb{R}^3 \\ Q_k|_{U_1}(u, q) &= \phi(u) (|u|, \text{Re } uq, \phi(u) \langle \text{Im } uq, \text{Im } q \rangle) \\ Q_k|_{U_2}(v, r) &= \phi(v) (|r|, \text{Re } r, \phi(v) \langle \text{Im } r, \text{Im } \bar{v}r \rangle) \end{aligned}$$

*is well-defined and has fibers that coincide with the orbits of  $G^\Lambda$ .*

*Proof.* To see that  $Q_k$  is well-defined, we will show

$$Q_k|_{U_1 \setminus \{0 \times \mathbb{S}^{b-1}\}} = Q_k|_{U_2 \setminus \{0 \times \mathbb{S}^{b-1}\}} \circ \Phi_k|_{U_1 \setminus \{0 \times \mathbb{S}^{b-1}\}}. \quad (2.2.1)$$

Since

$$\Phi_k(u, q) = \left( \frac{u}{|u|^2}, \left( \frac{u}{|u|} \right)^h q \left( \frac{u}{|u|} \right)^{-(h-1)} \right),$$

where  $k = 2h - 1$ , the left hand side of Equation (2.2.1) is

$$\begin{aligned} &Q_k|_{U_2 \setminus \{0 \times \mathbb{S}^{b-1}\}} \circ \Phi_k|_{U_1 \setminus \{0 \times \mathbb{S}^{b-1}\}}(u, q) = Q_k \left( \frac{u}{|u|^2}, \frac{u^h q u^{-(h-1)}}{|u|} \right) \\ &= \phi \left( \frac{u}{|u|^2} \right) \left( \left| \frac{u^h q u^{-(h-1)}}{|u|} \right|, \text{Re } \frac{u^h q u^{-(h-1)}}{|u|}, \phi \left( \frac{u}{|u|^2} \right) \left\langle \text{Im } \frac{u^h q u^{-(h-1)}}{|u|}, \text{Im } \frac{\bar{u}}{|u|^2} \frac{u^h q u^{-(h-1)}}{|u|} \right\rangle \right) \end{aligned} \quad (2.2.2)$$

To see that this is equal to  $Q_k|_{U_1 \setminus \{0 \times \mathbb{S}^{b-1}\}}(u, q)$ , we will simplify each coordinate separately. Before doing so we point out that

$$\begin{aligned} \frac{1}{|u|} \phi\left(\frac{u}{|u|^2}\right) &= \frac{1}{|u|} \frac{1}{\sqrt{1 + \frac{1}{|u|^2}}} \\ &= \frac{1}{\sqrt{|u|^2 + 1}} \\ &= \phi(u). \end{aligned} \tag{2.2.3}$$

So the first coordinate of the right hand side of Equation (2.2.2) is

$$\begin{aligned} \phi\left(\frac{u}{|u|^2}\right) \left| \frac{u^h q u^{-(h-1)}}{|u|} \right| &= \phi\left(\frac{u}{|u|^2}\right) \\ &= |u| \phi(u), \end{aligned} \tag{2.2.4}$$

and the second coordinate of the right hand side of Equation (2.2.2) is

$$\begin{aligned} \phi\left(\frac{u}{|u|^2}\right) \operatorname{Re} \frac{u^h q u^{-(h-1)}}{|u|} &= \phi\left(\frac{u}{|u|^2}\right) \operatorname{Re} \left( \frac{u q}{|u|} \right) \\ &= \frac{1}{|u|} \phi\left(\frac{u}{|u|^2}\right) \operatorname{Re}(u q) \\ &= \phi(u) \operatorname{Re}(u q), \text{ by Equation (2.2.3).} \end{aligned}$$

Finally, we have that the third coordinate of the right hand side of Equation (2.2.2) is

$$\begin{aligned} &\phi\left(\frac{u}{|u|^2}\right)^2 \left\langle \operatorname{Im} \frac{u^h q u^{-(h-1)}}{|u|}, \operatorname{Im} \frac{\bar{u}}{|u|^2} \frac{u^h q u^{-(h-1)}}{|u|} \right\rangle \\ &= \phi\left(\frac{u}{|u|^2}\right)^2 \left\langle \operatorname{Im} \frac{u^h q u^{-(h-1)}}{|u|}, \operatorname{Im} \frac{u^{h-1} q u^{-(h-1)}}{|u|} \right\rangle \\ &= \phi\left(\frac{u}{|u|^2}\right)^2 \frac{1}{|u|^2} \langle \operatorname{Im} u^{h-1}(u q) u^{-(h-1)}, \operatorname{Im} u^{h-1}(q) u^{-(h-1)} \rangle \\ &= \phi(u)^2 \langle \operatorname{Im} u q, \operatorname{Im} q \rangle, \text{ by Equation (2.2.3).} \end{aligned}$$

Combining the previous three displays with Equation (2.2.2) and the definition of  $Q_k|_{U_1}$ , we see that  $Q_k : S_k^{2b-2} \longrightarrow Q_k(S_k^{2b-2}) \subsetneq \mathbb{R}^3$  is well-defined.

To see that  $Q_k|_{U_1}$  is constant on each orbit of  $G^\Lambda$ , we use the fact that  $G^\Lambda$  acts by isometries and commutes with conjugation to get

$$\begin{aligned} \operatorname{Re} g(u) g(q) &= \langle g(u), \overline{g(q)} \rangle \\ &= \langle g(u), g(\bar{q}) \rangle \\ &= \langle u, \bar{q} \rangle \\ &= \operatorname{Re}(u q). \end{aligned}$$

We also have

$$\begin{aligned}
& \langle \operatorname{Im}(g(u)g(q)), \operatorname{Im}g(q) \rangle \\
&= \langle \operatorname{Re}(g(u))\operatorname{Im}g(q) + \operatorname{Re}(g(q))\operatorname{Im}g(u) + \operatorname{Im}g(u)\operatorname{Im}g(q), \operatorname{Im}g(q) \rangle \\
&= \langle \operatorname{Re}(u)\operatorname{Im}g(q) + \operatorname{Re}(q)\operatorname{Im}g(u), \operatorname{Im}g(q) \rangle \\
&= \langle g(\operatorname{Re}(u)\operatorname{Im}(q) + \operatorname{Re}(q)\operatorname{Im}(u)), g(\operatorname{Im}(q)) \rangle \\
&= \langle \operatorname{Re}(u)\operatorname{Im}(q) + \operatorname{Re}(q)\operatorname{Im}(u), \operatorname{Im}(q) \rangle \\
&= \langle \operatorname{Re}(u)\operatorname{Im}(q) + \operatorname{Re}(q)\operatorname{Im}(u) + \operatorname{Im}uq\operatorname{Im}q, \operatorname{Im}(q) \rangle \\
&= \langle \operatorname{Im}(uq), \operatorname{Im}q \rangle.
\end{aligned}$$

Since  $|g(u)| = |u|$  and  $\phi(gu) = \phi(u)$ , it follows that

$$Q_k|_{U_1} \begin{pmatrix} g(u) \\ g(q) \end{pmatrix} = Q_k|_{U_1} \begin{pmatrix} u \\ q \end{pmatrix}.$$

Combining this with

$$\begin{aligned}
Q_k|_{U_2} g \begin{pmatrix} 0 \\ r \end{pmatrix} &= (1, \operatorname{Re}(r), 0) \\
&= Q_k|_{U_2} \begin{pmatrix} 0 \\ r \end{pmatrix},
\end{aligned}$$

it follows that  $Q_k$  is constant on each orbit of  $G^\Lambda$ .

On the other hand, if

$$Q_k|_{U_1}(u_1, q_1) = Q_k|_{U_1}(u_2, q_2),$$

then

$$\phi(u_1)|u_1| = \phi(u_2)|u_2|, \quad (2.2.5)$$

$$\phi(u_1)^2 \langle \operatorname{Im}(u_1q_1), q_1 \rangle = \phi(u_2)^2 \langle \operatorname{Im}(u_2q_2), q_2 \rangle, \text{ and} \quad (2.2.6)$$

$$\phi(u_1)\operatorname{Re}u_1q_1 = \phi(u_2)\operatorname{Re}u_2q_2. \quad (2.2.7)$$

Equation (2.2.5) implies that  $|u_1| = |u_2|$  and  $\phi(u_1) = \phi(u_2)$ . So

$$\begin{aligned}
\operatorname{Re}(u_1) &= \operatorname{Re}(u_1) \langle q_1, q_1 \rangle \\
&= \langle (\operatorname{Re}(u_1) + \operatorname{Im}(u_1))q_1, q_1 \rangle, \text{ since } \operatorname{Re}(q_1) = 0 \\
&= \langle u_1q_1, q_1 \rangle \\
&= \langle \operatorname{Im}(u_1q_1), q_1 \rangle, \text{ since } \operatorname{Re}(q_1) = 0 \\
&= \langle \operatorname{Im}(u_2q_2), q_2 \rangle, \text{ by Equation (2.2.6) and the fact that } \phi(u_1) = \phi(u_2) \\
&= \operatorname{Re}(u_2)
\end{aligned}$$



and

$$\begin{aligned}
\langle \operatorname{Im}(u_1), q_1 \rangle &= -\langle u_1, \bar{q}_1 \rangle, \text{ since } \operatorname{Re}(q_1) = 0 \\
&= -\operatorname{Re} u_1 q_1 \\
&= -\operatorname{Re} u_2 q_2, \text{ by Equation 2.2.7 and the fact that } \phi(u_1) = \phi(u_2) \\
&= -\langle u_2, \bar{q}_2 \rangle \\
&= \langle \operatorname{Im}(u_2), q_2 \rangle.
\end{aligned}$$

Together with  $|u_1| = |u_2|$  and the fact that  $q_1$  and  $q_2$  are imaginary, the previous two displays imply that  $\begin{pmatrix} u_1 \\ q_1 \end{pmatrix}$  and  $\begin{pmatrix} u_2 \\ q_2 \end{pmatrix}$  are in the same orbit.

Finally suppose that

$$Q_k|_{U_2}(0, r_1) = Q_k|_{U_2}(0, r_2).$$

Then

$$(1, \operatorname{Re}(r_1), 0) = (1, \operatorname{Re}(r_2), 0).$$

Since we also have that  $|r_1| = |r_2| = 1$ , it follows that  $(0, r_1)$  and  $(0, r_2)$  are in the same  $G^\Lambda$ -orbit.  $\square$

**Key Lemma 2.3.** *Let  $Q_s$  be as in Lemma 2.1.*

1. *There is a well-defined surjective map*

$$\tilde{Q}_k : S_k^{2b-2} \rightarrow \mathbb{S}^{2b-2}/G^\Lambda$$

*whose fibers coincide with the orbits of the  $G^\Lambda$  action on  $S_k^{2b-2}$ .*

2. *The orbit types of  $p \in S_k^{2b-2}$  and  $Q_s^{-1}(\tilde{Q}_k(p))$  coincide.*

3. *For  $p \in S_k^{2b-2}$  and any  $q \in Q_s^{-1}(\tilde{Q}_k(p))$  the isotropy representation of  $G_p^\Lambda$  and  $G_q^\Lambda$  are equivalent.*

*In particular,  $\mathbb{S}^{2b-2}/G^\Lambda$  and  $S_k^{2b-2}/G^\Lambda$  are equivalent orbit spaces.*

*Proof.* Motivated by [7, 20], we let  $h_1, h_2 : \Lambda \times \mathbb{S}^{b-2} \rightarrow \mathbb{S}^{2b-2}$  be given by

$$\begin{aligned}
h_1(u, q) &= \begin{pmatrix} uq \\ q \end{pmatrix} \phi(u) \text{ and} \\
h_2(v, r) &= \begin{pmatrix} r \\ \bar{v}r \end{pmatrix} \phi(v).
\end{aligned}$$

We claim that  $Q_s$  and  $Q_k$  are related by

$$Q_k = \begin{cases} Q_s \circ h_1 & \text{on } U_1 \\ Q_s \circ h_2 & \text{on } U_2 \end{cases}. \quad (2.3.1)$$

Indeed,

$$\begin{aligned}
Q_s \circ h_1(u, q) &= Q_s \left( \begin{pmatrix} uq \\ q \end{pmatrix} \right) \phi(u) \\
&= \phi(u) (|u|, \operatorname{Re} uq, \phi(u) \langle \operatorname{Im} uq, \operatorname{Im} q \rangle) \\
&= Q_k(u, q)
\end{aligned} \quad (2.3.2)$$

and

$$\begin{aligned}
Q_s \circ h_2(v, r) &= Q_s \left( \frac{r}{\bar{v}r} \right) \phi(v) \\
&= \phi(v) (|r|, \operatorname{Re}(r), \phi(v) \langle \operatorname{Im} r, \operatorname{Im} \bar{v}r \rangle) \\
&= Q_k(v, r),
\end{aligned}$$

proving Equation (2.3.1).

Since  $h_1(\Lambda \times \mathbb{S}^{b-2}) \cup h_2(\Lambda \times \mathbb{S}^{b-2}) = \mathbb{S}^{2b-2}$ , Equation (2.3.1) implies that  $Q_k(S_k^{2b-2}) = Q_s(\mathbb{S}^{2b-2})$ ; so setting  $\tilde{Q}_k = Q_k$  gives a well-defined surjective map

$$\tilde{Q}_k : S_k^{2b-2} \rightarrow \mathbb{S}^{2b-2}/G^\Lambda,$$

and Part 1 is proven. Parts 2 and 3 follow from the observation that  $h_1$  and  $h_2$  are  $G^\Lambda$ -equivariant embeddings.  $\square$

Since the antipodal map  $A : \mathbb{S}^{2b-2} \rightarrow \mathbb{S}^{2b-2}$  and the involution

$$T : S_k^{2b-2} \rightarrow S_k^{2b-2}$$

from page 4, commute with the  $G^\Lambda$ -actions (0.0.1), (0.0.2) and (1.0.6), they induce well-defined  $\mathbb{Z}_2$ -actions on our orbit space  $Q_s(\mathbb{S}^{2b-2}) = Q_e(S_e^{2b-2}) =$

$$\left\{ (x, y, z) \mid x \in [0, 1], y \in [-x, x], z \in \left[ -\sqrt{(x^2 - y^2)(1 - x^2)}, \sqrt{(x^2 - y^2)(1 - x^2)} \right] \right\}.$$

A simple calculation shows that the two  $\mathbb{Z}_2$ -actions on  $Q_s(\mathbb{S}^{2b-2})$  coincide and are given by

$$(x, y, z) \mapsto (x, -y, z).$$

Since quotient maps of isometric group actions preserve lower curvature bounds,  $\mathbb{S}^{2b-2}/(SO(3) \times \mathbb{Z}_2)$  has curvature  $\geq 1$  ([4]). Therefore, Theorem A follows from Theorem B and Key Lemma 2.3.

### 3. SOME CLOSING REMARKS

In the same paper, Hirsch and Milnor also constructed exotic  $\mathbb{R}P^5$ s,  $P_k^5$ s. The Davis action also descends to the  $P_k^5$ s where they commute with an  $SO(2)$ -action. The combined  $SO(2) \times SO(3)$ -action on the  $P_k^5$ s is by cohomogeneity one. Dearricott and Grove-Ziller observed that since these cohomogeneity one actions have codimension 2 singular orbits, Theorem E of [9] implies that they admit invariant metrics of nonnegative curvature.

Octonionically, the Hirsch-Milnor construction yields closed 13-manifolds,  $P_k^{13}$ , that are homotopy equivalent to  $\mathbb{R}P^{13}$ . Their proof that the  $P_k^5$ s are not diffeomorphic to  $\mathbb{R}P^5$  breaks down, since in contrast to dimension 6, there is an exotic 14-sphere; however, Chenxu He has informed us that some of the  $P_k^{13}$ s are in fact exotic ([11]).

The Davis construction yields a cohomogeneity one action of  $SO(2) \times G_2$  on the  $P_k^{13}$ s, only now one of the singular orbits has codimension 6. So we cannot apply Theorem E of [9]. Moreover, there are cohomogeneity one manifolds that do not admit invariant metrics with nonnegative curvature ([8, 10]). On the other hand, by the main theorem of [16], every cohomogeneity one manifold admits an invariant metric with almost nonnegative curvature.

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